Eur. Phys. J. C 44, 447–457 (2005)
Digital Object Identifier (DOI) 10.1140/epjc/s2005-02374-x

THE EUROPEAN
PHYSICAL JOURNAL C

High scale perturbative gauge coupling in R-parity conserving SUSY $SO(10)$ with longer proton lifetime

M.K. Parida[a], B.D. Cajee

Physics Department, North-Eastern Hill University, Shillong 793022, India

Received: 1 July 2005 / Revised version: 21 August 2005 /
Published online: 11 October 2005 –

Abstract. It is well known that in single step breaking of R-parity conserving SUSY $SO(10)$ that needs the Higgs representations $\underline{126}\oplus\overline{\underline{126}}$, the GUT gauge coupling violates the perturbative constraint at mass scales a few times larger than the GUT scale. Therefore, if the $SO(10)$ gauge coupling is to remain perturbative up to the Planck scale ($\equiv 2\times10^{18}$ GeV), the scale M_U of the GUT symmetry breaking is to be bounded from below. The bound depends upon specific Higgs representations used for $SO(10)$ symmetry breaking but, as we find, cannot be lower than 1.5×10^{17} GeV. In order to obtain such a high unification scale we propose a two-step $SO(10)$ breaking through $SU(2)_L\times SU(2)_R\times U(1)_{B-L}\times SU(3)_C$ ($g_{2L}\neq g_{2R}$) intermediate gauge symmetry. We estimate the potential threshold and gravitational corrections to the gauge coupling running and show that they can make the picture of perturbative gauge coupling running consistent at least up to the Planck scale. We also show that when $SO(10)\to G_{2213}$ by the Higgs representations $\underline{210}\oplus\underline{54}$, gravitational corrections alone with negligible threshold effects may guarantee such perturbative gauge coupling. The lifetime of the proton is found to increase by nearly 6 orders over the present experimental limit for $p\to e^+\pi^0$. For the proton decay mediated by a dim = 5 operator a wide range of lifetimes is possible, extending from the current experimental limit up to values 2–3 orders longer.

1 Introduction

In spite of its astounding success the non-supersymmetric standard model (SM) suffers from the well known gauge hierarchy problem. It fails to explain the available data on neutrino masses and mixings and also fails to exhibit unification of the three known gauge couplings at higher scales. One compelling reason to solve the gauge hierarchy problem is to go beyond the SM through weak scale SUSY as in the minimal supersymmetric standard model (MSSM) [1]. The MSSM has the added virtues that, in addition to explaining the origin of electroweak symmetry breaking, it provides a candidate for dark matter of the universe. If the SM fermion representations are extended by the addition of one right-handed neutrino per generation and the corresponding extension is made in MSSM, the model can account for neutrino masses and mixings through seesaw mechanisms [2–4].

Another amazing aspect of MSSM has been noted to be the unification of the three gauge couplings of disparate strengths and origins when extrapolated to as high a scale as $M_U = 2\times10^{16}$ GeV [5]. However, the meeting of the three gauge couplings can be truly termed the grand unification [6, 7] of the three basic forces of nature provided the merged coupling constants evolve as a single gauge coupling at higher scales and some simple ansatz for this has been hypothesized through SUSY GUTs such as $SU(5)$, $SO(10)$, E_6, and a number of other ones [7–9].

While R-parity violation as an automatic consequence of MSSM spoils the predictive power of supersymmetric theories, an additional elegant feature of SUSY $SO(10)$ breaking down to MSSM is its potentiality to conserve R-parity. As the minimal left–right symmetric GUT $SO(10)$ contains the maximal subgroup $SU(2)_L\times SU(2)_R\times SU(4)_C$ of Pati–Salam [6] which in turn contains $SU(2)_L\times U(1)_R\times SU(4)_C$, $SU(2)_L\times SU(2)_R\times U(1)_{B-L}\times SU(3)_C$ ($\equiv G_{2213}$), $SU(2)_L\times U(1)_R\times U(1)_{B-L}\times SU(3)_C$ and $SU(2)_L\times U(1)_Y\times SU(3)_C$ ($\equiv G_{213}\equiv$ SM) as its subgroups [10]. Thus, subject to the consistency with the renormalization group constraints, SUSY $SO(10)$ gauge symmetry may break to the SM gauge group directly in one step or through an intermediate gauge symmetry to the MSSM [11, 20]. In addition to other superheavy representations needed to implement the GUT symmetry breaking, two different popular choices of Higgs representations being extensively used to obtain the MSSM from SUSY$SO(10)$ are $\underline{16}\oplus\overline{\underline{16}}$ and $\underline{126}\oplus\overline{\underline{126}}$. While the first choice violates R-parity, the second one conserves it. The Higgs representations $\underline{126}\oplus\overline{\underline{126}}$ in SUSY $SO(10)$ have been found to solve a number of problems on fermion masses through renormalizable interactions. To cite a few, it rectifies the bad $SU(5)$ mass relation in the right direction in$SO(10)$ to yield $m_\mu = 3m_s$. It attributes large

[a] e-mail: mparida@sancharnet.in

atmospheric neutrino mixing to b–τ unification and accommodates the masses and mixings of three neutrino flavors, in addition to the observed masses and mixings of all other fermions, via the Type II seesaw mechanism [12–15, 21]. However, because of the large contribution to the β-function coefficient of the gauge coupling evolution, the presence of $\underline{126} \oplus \overline{\underline{126}}$ in R-parity conserving SUSY$SO(10)$, in addition to other Higgs representations, violates the perturbative constraint on the GUT gauge coupling ($\alpha_G < 1$) even at mass scales a few times larger than the GUT scale ($= 2 \times 10^{16}$ GeV). Although there might be deeper reasons to believe that the R-conserving SUSY at such scales could be non-perturbative, it is desirable to have a perturbative theory at least up to the compactification scale ($M_{\rm CS} \simeq 10^{17}$ GeV) or the Planck scale ($M_{\rm Pl} = 2 \times 10^{18}$ GeV).

Proton decay is a necessary prediction of a number of GUTs including $SU(5)$ and $SO(10)$. The decay mode $p \to e^+\pi^0$ common to both SUSY and non-SUSY GUTs is mediated by superheavy gauge bosons carrying fractional charges and the corresponding effective Lagrangian has a dim = 6 operator. In SUSY GUTs, the superpartners of fermions and heavy color triplets of Higgs bosons give rise to new decay modes such as $p \to K^+\overline{\nu}_\mu$, $p \to K^+\overline{\nu}_\tau$, and other ones. The mediation of the heavy superpartner leads to a dim = 5 operator in the effective Lagrangian for these supersymmetric decay modes. Recent experimental measurements provide improved limits on the lifetimes for both these types of decay modes,

$$\tau\left(p \to e^+\pi^0\right) \geq 4 \times 10^{33}\,\text{years}, \quad (1)$$

$$\tau(p \to K^+\overline{\nu}_\tau) \geq 2.2 \times 10^{33}\,\text{years}. \quad (2)$$

While (1) gives the bound $M_U \geq 5.6 \times 10^{15}$ GeV, (2) yields the limit on the superheavy color triplet higgsino mass $M_{T_{\tilde{C}}} \geq 10^{17}$ GeV. Although this has been treated as a severe constraint on SUSY $SU(5)$ [16], easier methods have been suggested to evade it [17, 18]. In R-parity conserving SUSY $SO(10)$ other interesting suggestions have been made to increase the proton lifetime of the supersymmetric decay mode through specific Yukawa textures, but in this case the GUT gauge coupling remains perturbative only up to $\mu = \text{few} \times 2 \times 10^{16}$ GeV [19].

In this paper we show that with the similar choices of the Higgs representations as in the single step breakings of R-parity conserving SUSY $SO(10)$, when the GUT gauge symmetry is allowed to break down to MSSM through the G_{2213}-intermediate gauge symmetry investigated recently [20], perturbative GUT gauge coupling is ensured at least up to the Planck scale due to threshold and gravitational corrections. Although in this paper we have addressed the issue of perturbative gauge coupling up to the reduced Planck scale ($= 2 \times 10^{18}$ GeV), we have checked that our method also works even if we use the Planck scale as $M_{\rm Pl} \simeq 1.2 \times 10^{19}$ GeV according to the definition of the Particle Data Group. The realization of perturbative grand unification in R-parity conserving $SO(10)$ which has not been possible otherwise is demonstrated for the first time in this paper. Other new contributions of the present paper compared to [20] are derivations of gravitational corrections in the presence of Higgs representations $\underline{54}$ and $\underline{210} \oplus \underline{54}$ which contribute to $SO(10)$ breaking near the GUT scale. Combining perturbative criteria with R-parity conservation in SUSY $SO(10)$ we obtain lower bounds on the unification scale in different cases. A very significant increase of proton lifetimes is obtained leading to a greater stability of the particle.

In Sect. 2, we discuss the origin of high scale violation of perturbation theory in SUSY $SO(10)$. In Sect. 3 we discuss analytically the threshold and gravitational corrections. In Sect. 4 we show how these corrections elevate the unification scale so as to satify the perturbative constraint on the GUT gauge coupling at least up to the Planck scale. In Sect. 5 we discuss the increase in proton lifetimes in different cases. A summary and our conclusions are stated in Sect. 6.

2 Perturbative constraint and lower bounds on unification scale

With R-parity conservation a minimal $SO(10)$ model having 26 parameters has been identified to be the one with Higgs representations: $\underline{210} \oplus \underline{126} \oplus \overline{\underline{126}} \oplus \underline{10}$ [31] for which a very interesting method of proton lifetime increase has been suggested [19]. In order to account for neutrino masses and mixings in SUSY $SO(10)$ through Type II seesaw dominance, the realistic symmetry breaking pattern has been shown to require $\underline{210} \oplus \underline{54} \oplus \underline{126} \oplus \overline{\underline{126}} \oplus \underline{10}$ [21] where both $\underline{210}$ and $\underline{54}$ are present. We will show that in this case with G_{2213}-intermediate breaking gravitational corrections alone may be sufficient to guarantee perturbative gauge coupling at higher scales. But, in the single step breaking scenario above the GUT scale, not only these two models but also other variants of R-parity conserving SUSY $SO(10)$ violate perturbation theory even at mass scales $\mu = \text{few} \times 2 \times 10^{16}$ GeV whenever the Higgs representations $\underline{126} \oplus \overline{\underline{126}}$ are present in the model.

Above the GUT scale ($\mu > M_U$) the GUT fine structure constant $\alpha_G(\mu) = \frac{g_G^2(\mu)}{4\pi}$, where g_G is the GUT coupling, evolves at one-loop level as

$$\frac{1}{\alpha_G(\mu)} = \frac{1}{\alpha_G(M_U)} - \frac{a}{2\pi}\ln\frac{\mu}{M_U} \quad (3)$$

The β-function coefficient in (3) consists of gauge, matter and Higgs contributions,

$$a = a_{\rm gauge} + a_{\rm matter} + a_{\rm Higgs}. \quad (4)$$

The gauge bosons of $SO(10)$ in the adjoint representation $\underline{45}$, three generations of matter in the spinorial representations $\underline{16}$, and their superpartners contribute as

$$a_{\rm gauge} = -24, \quad a_{\rm matter} = 6, \quad (5)$$

The Higgs contributions of different $SO(10)$ irreducible representations are shown in Table 1.

Noting that $a_{\rm gauge} + a_{\rm matter} = -18$, use of (5) in (3) and (4) gives at $\mu = \Lambda > M_U$

$$\frac{1}{\alpha_G(\Lambda)} = \frac{1}{\alpha_G(M_U)} + \frac{18}{2\pi}\ln\frac{\Lambda}{M_U} - \frac{a_{\rm Higgs}}{2\pi}\ln\frac{\Lambda}{M_U}. \quad (6)$$

Table 1. Contribution of Higgs representations to the SUSY $SO(10)$ β-function coefficient for the GUT gauge coupling evolution

Rep.	a_{Higgs}	Rep.	a_{Higgs}
$\underline{10}$	1	$\underline{45} \oplus \underline{16} \oplus \overline{\underline{16}} \oplus \underline{10}$	13
$\underline{54}$	12	$\underline{54} \oplus \underline{45} \oplus \underline{16} \oplus \overline{\underline{16}} \oplus \underline{10}$	25
$\underline{120}$	28	$\underline{210} \oplus \underline{16} \oplus \overline{\underline{16}} \oplus \underline{10}$	61
$\underline{16}$	2	$\underline{45} \oplus \underline{126} \oplus \overline{\underline{126}} \oplus \underline{10}$	79
$\underline{45}$	8	$\underline{210} \oplus \underline{126} \oplus \overline{\underline{126}} \oplus \underline{10}$	127
$\underline{126}$	35	$\underline{54} \oplus \underline{45} \oplus \underline{126} \oplus \overline{\underline{126}} \oplus \underline{10}$	91
$\underline{210}$	56	$\underline{210} \oplus \underline{54} \oplus \underline{126} \oplus \overline{\underline{126}} \oplus \underline{10}$	139

If the gauge coupling constant encounters a Landau pole at Λ, $\alpha_G(\Lambda) \longrightarrow \infty$, and (6) leads to

$$a_{\text{Higgs}} \leq 18 + \frac{2\pi}{\ln\left(\frac{\Lambda}{M_U}\right)} \times \frac{1}{\alpha_G(M_U)}. \quad (7)$$

On the other hand the perturbative condition

$$\alpha_G(\Lambda) \leq 1. \quad (8)$$

leads to the constraint

$$a_{\text{Higgs}} \leq 18 + \frac{2\pi}{\ln\left(\frac{\Lambda}{M_U}\right)} \left[\frac{1}{\alpha_G(M_U)} - 1\right]. \quad (9)$$

In the single step breakings of all SUSY GUTs $M_U \simeq 2 \times 10^{16}$ GeV, $\alpha_G(M_U)^{-1} \simeq 25$, and the upper bound defined by inequality (9) has been estimated [22].

For SUSY $SO(10)$ with $\underline{45} \oplus \underline{16} \oplus \overline{\underline{16}} \oplus \underline{10}$, $a_{\text{Higgs}} = 13$ and the perturbative constraint remains valid for higher scales, and perturbative grand unification is guaranteed at least up to the Planck scale [22]. However, for minimal $SO(10)$ with $\underline{210} \oplus \underline{126} \oplus \overline{\underline{126}} \oplus \underline{10}$, $a_{\text{Higgs}} = 127$ and the perturbation theory cannot be guaranteed to hold up to the Planck scale in the grand-desert model. Thus, in the single step breaking of SUSY $SO(10)$ to MSSM, whenever larger Higgs representations like $\underline{126} \oplus \overline{\underline{126}}$ are used to break the $SU(2)_R \times U(1)_{B-L} \subset SO(10)$ or $SU(2)_R \times SU(4)_C \subset SO(10)$, leading to the seesaw mechanism and Majorana neutrino masses, the large contribution to the Dynkin indices violates perturbation theory at Λ = few $\times\, 2 \times 10^{16}$ GeV. This has led to the investigations of perturbative grand unification of $SO(10)$ through the use of the Higgs representations $\underline{16} \oplus \overline{\underline{16}}$ instead of $\underline{126} \oplus \overline{\underline{126}}$ in the supergrand-desert scenario [22].

It is clear that in R-parity conserving SUSY $SO(10)$ the Higgs contribution to the β-function coefficient for the gauge coupling evolution satisfies $a_{\text{Higgs}} > 71$. Noting that $\alpha_G(M_U) \simeq 0.043$ and demanding that perturbative condition is satisfied up to $\Lambda = M_{\text{Pl}} = 2 \times 10^{18}$ GeV, then the inequality (9) gives the lower bound

$$M_U > 10^{17}\,\text{GeV}.$$

This lower bound on the unification scale has to be satisfied in any R-parity conserving SUSY $SO(10)$ if the GUT gauge coupling is to remain perturbative up to the Planck scale. It is interesting to note that this lower bound accidentally matches the higgsino mass limit obtained from the current experimental limit of the proton lifetime for $p \to K^+\overline{\nu}_{\mu,\tau}$.

In the four specific examples of Higgs representations shown in Table 1 which correspond to R-parity conservation, the Higgs contributions to the β-function coefficients in the respective cases and the inequality (9) give different values of lower bounds on the unification scale. In particular for the choices of the Higgs representations
(I) $\underline{210} \oplus \underline{126} \oplus \overline{\underline{126}} \oplus \underline{10}$,
(II) $\underline{54} \oplus \underline{45} \oplus \underline{126} \oplus \overline{\underline{126}} \oplus \underline{10}$,
(III) $\underline{210} \oplus \underline{54} \oplus \underline{126} \oplus \overline{\underline{126}} \oplus \underline{10}$, and
(IV) $\underline{45} \oplus \underline{126} \oplus \overline{\underline{126}} \oplus \underline{10}$, the lower bounds on the unification scale turn out to be $M_U = 5.8 \times 10^{17}$ GeV, $M_U = 3 \times 10^{17}$ GeV, $M_U = 6.25 \times 10^{17}$ GeV, and $M_U = 1.5 \times 10^{17}$ GeV, respectively. Thus the smallest lower bound corresponds to the one for the Higgs representation $\underline{45} \oplus \underline{126} \oplus \overline{\underline{126}} \oplus \underline{10}$ as expected with the minimal contribution $a_{\text{Higgs}} = 79$. These lower bounds suggest that if the perturbative criteria on the GUT gauge coupling is to be satisfied, the unification scale has to be elevated by at least 1 order compared to the conventional value. Further, the perturbative constraint has the implication that, in R-conserving SUSY $SO(10)$, the larger is the Higgs contribution to the β-function coefficient, the greater must be the unification scale. The lower bounds are to be satisfied irrespective of the$SO(10)$ breaking to MSSM through a single step or through an intermediate gauge symmetry.

In the next section we show how the presence of $SU(2)_L \times SU(2)_R \times U(1)_{B-L} \times SU(3)_C$-intermediate gauge symmetry at higher scales yields perturbative $SO(10)$ up to the Planck scale even if we use the Higgs representations $\underline{126} \oplus \overline{\underline{126}}$ with or without $\underline{210}$ or other Higgs representations such as $\underline{54}$ and $\underline{45}$ for high scale breaking of SUSY $SO(10)$.

3 Threshold and gravitational corrections on mass scales

It is clear from (9) that, if in a specific GUT scenario the unification scale M_U can be closer to the Planck or the compactification scale than in the single step breaking case, the contribution of the Higgs representation to the RHS of (9) can be larger without violating the inequality. In [20] the intermediate G_{2213} breaking in SUSY $SO(10)$ was investigated, and we have

$$SO(10) \times \text{SUSY} \xrightarrow[M_U]{\Phi_U} G_{2213} \times \text{SUSY} \quad (10)$$
$$\xrightarrow[M_I]{\underline{126} \oplus \overline{\underline{126}}} G_{213} \times \text{SUSY} \xrightarrow[M_Z]{\underline{10}} U(1)_{em} \times SU(3)_C,$$

where the Higgs representations responsible for the GUT symmetry breaking were chosen as $\Phi_U \equiv \underline{210}$, or $\underline{54} \oplus \underline{45}$ which also break D-parity at the GUT scale while permitting the left–right asymmetric gauge group G_{2213} ($g_{2L} \neq g_{2R}$) to survive down to the intermediate scale [23]. In such an R-parity conserving symmetry breaking chain quite significant threshold corrections arising out of spreading of masses around the intermediate scale and the GUT scale and gravitational corrections arising out of dim = 5 operators induced by the Planck or the compactification scales [24–27] were noted. In this section we estimate these effects in detail in order to explore the possibility of increasing M_U which is necessary for the existence of a perturbative gauge coupling at higher scales. While the gravitational corrections originating from the dim = 5 operator due to $\underline{210}$ was investigated in [20], in this work we investigate the corresponding effects due to $\underline{54}$ and $\underline{210} \oplus \underline{54}$ while studying the threshold effects of the latter. The evolution of gauge couplings in the two different mass ranges is expressed as

$$\frac{1}{\alpha_i(M_Z)} = \frac{1}{\alpha_i(M_I)} + \frac{a_i}{2\pi} \ln \frac{M_I}{M_Z} + \theta_i - \Delta_i, \quad i = 1Y, 2L, 3C, \tag{11}$$

$$\frac{1}{\alpha_i(M_I)} = \frac{1}{\alpha_i(M_U)} + \frac{a'_i}{2\pi} \ln \frac{M_U}{M_I} + \theta'_i - \Delta'_i - \Delta_i^{(\mathrm{gr})}, \quad i = 2L, 2R, BL, 3C, \tag{12}$$

where the second, third, and the fourth terms in the RHS of (11) and (12) represent one-loop, two-loop, threshold, and gravitational corrections, respectively [20]. In (11) and (12) $a_Y = 33/5, a_{2L} = 1, a_{3C} = a'_{3C} = -3, a'_{2L} = 1, a'_{2R} = 5$ and $a'_{BL} = 15$. The two-loop coefficients (b_{ij}) below the intermediate scale and (b'_{ij}) above the intermediate scale have been obtained in [20]. Below M_I the presence of G_{213} in MSSM gives

$$b_{ij} = \begin{pmatrix} \frac{199}{25} & \frac{27}{5} & \frac{88}{5} \\ \frac{9}{5} & 25 & 24 \\ \frac{11}{5} & 9 & 14 \end{pmatrix}, \; i,j = 1Y, 2L, 3C. \tag{13}$$

Above the intermediate scale, the two-loop beta-function coefficients in the presence of SUSY G_{2213} symmetry are

$$b'_{ij} = \begin{pmatrix} 25 & 3 & 3 & 24 \\ 3 & 73 & 27 & 24 \\ 9 & 81 & 61 & 8 \\ 9 & 9 & 1 & 14 \end{pmatrix}, \quad i,j = 2L, 2R, BL, 3C. \tag{14}$$

These coefficients occur in two-loop contributions represented by θ_i and θ'_i in the two mass ranges;

$$\theta_i = \frac{1}{4\pi} \sum_j B_{ij} \ln \frac{\alpha_j(M_I)}{\alpha_j(M_Z)},$$

$$\theta'_i = \frac{1}{4\pi} \sum_j B'_{ij} \ln \frac{\alpha_j(M_U)}{\alpha_j(M_I)},$$

$$B_{ij} = \frac{b_{ij}}{a_j}, \qquad B'_{ij} = \frac{b'_{ij}}{a'_j}. \tag{15}$$

For the sake of simplicity we have neglected the Yukawa contributions to two-loop effects on gauge couplings. While the functions Δ_i include threshold effects at M_Z and M_I with

$$\Delta_i = \Delta_i^{(Z)} + \Delta_i^{(I)},$$

the Δ'_i include threshold effects at M_U.

It may be recalled that although in non-supersymmetric gauge theories threshold effects contain both constant terms as well as logarithmic terms, it was noted in [32] that the constant terms are absent in supersymmetric threshold corrections.

In the presence of G_{2213}-intermediate symmetry the particle spectra of Higgs scalars, fermions, gauge bosons, and their superpartners with masses lighter than M_U are the same in all four cases considered in this paper. Then, under the assumption that all superheavy particles with masses larger than M_U decouple from the Lagrangian, the contributions to the renormalization group evolutions of gauge and Yukawa couplings up to two loops below M_U are identical in all the four cases,

Case (I): $\underline{210} \oplus \underline{126} \oplus \underline{\overline{126}} \oplus \underline{10}$,
Case (II): $\underline{54} \oplus \underline{45} \oplus \underline{126} \oplus \underline{\overline{126}} \oplus \underline{10}$,
Case (III): $\underline{210} \oplus \underline{54} \oplus \underline{126} \oplus \underline{\overline{126}} \oplus \underline{10}$, and
Case (IV): $\underline{45} \oplus \underline{126} \oplus \underline{\overline{126}} \oplus \underline{10}$.

However, the GUT threshold and gravitational effects expressed through Δ'_i and $\Delta_i^{(\mathrm{gr})}$, respectively, differ from one choice of representation to another.

3.1 Threshold effects with effective mass parameters

We follow the method of effective mass parameters due to Carena, Pokorski and Wagner [28] to estimate threshold effects which have been also utilised to study such effects in SUSY $SU(5)$ by introducing two sets of effective mass parameters, one set for the SUSY threshold and the other set for the GUT threshold [29]. In [20] their effects have been examined on SUSY $SO(10)$ with G_{2213}-intermediate symmetry by defining one set of effective mass parameters for each threshold. Although these parameters at the weak scale SUSY threshold have been approximately estimated [28, 29], no such estimations are available for higher thresholds and they would be assumed to deviate at most by a factor 6 (1/6) from the corresponding scales. Following the standard procedure, the effective mass parameters are defined through the following relations:

$$\Delta_i^Z = \sum_\alpha \frac{b_i^\alpha}{2\pi} \ln \frac{M_\alpha}{M_Z} = \frac{b_i}{2\pi} \ln \frac{M_i}{M_Z}, \quad i = 1Y, 2L, 3C;\; \mu = M_Z; \tag{16}$$

$$\Delta_i^I = \sum_\alpha \frac{b_i'^\alpha}{2\pi} \ln \frac{M'_\alpha}{M_I} = \frac{b'_i}{2\pi} \ln \frac{M'_i}{M_I}, \quad i = 1Y, 2L, 3C;\; \mu = M_I; \tag{17}$$

$$\Delta' = \Delta_i^U = \sum_\alpha \frac{b_i''^\alpha}{2\pi} \ln \frac{M''_\alpha}{M_U} = \frac{b''_i}{2\pi} \ln \frac{M''_i}{M_U}, \quad i = 2L, 2R, BL, 3C;\; \mu = M_U; \tag{18}$$

where α refers to the actual G_{213} submultiplet near $\mu = M_Z$, M_I or G_{2213} submultiplet near $\mu = M_U$ and M_α, M'_α or M''_α refer to the actual component masses. The three sets of effective mass parameters are $M_{\rm i}$, $M'_{\rm i}$, and $M''_{\rm i}$. The coefficients $b'_i = \sum b'^{\alpha}_i$ and $b''_i = \sum b''^{\alpha}_i$ have been defined in (16)–(18) following [20,28]. The numbers b^α_i and b'^α_i refer to the contributions of the multiplet α to the β-functions of $U(1)_Y$, $SU(2)_L$, and $SU(3)_C$ gauge couplings. Similarly b''^α_i refers to the contributions of the multiplet α to the β-functions of $U(1)_Y$, $SU(2)_L$, $SU(2)_R$, $SU(3)_C$, and $U(1)_{B-L}$ gauge couplings [20].

The threshold effects on the mass scales M_I and M_U are then expressed in the form

$$\begin{aligned}\Delta \ln \frac{M_I}{M_Z} &= a \ln \frac{M''_{2R}}{M_U} + b \ln \frac{M''_{BL}}{M_U} + c \ln \frac{M''_{2L}}{M_U} \\ &\quad + d \ln \frac{M''_{3C}}{M_U} + e \ln \frac{M'_{1Y}}{M_I} - 1.56, \\ \Delta \ln \frac{M_U}{M_Z} &= a' \ln \frac{M''_{2L}}{M_U} + b' \ln \frac{M''_{3C}}{M_U} + 0.105, \end{aligned} \quad (19)$$

where the numerical values are due to the weak scale SUSY threshold effects. The values of the parameters computed for the four different cases are

Case (I): $\underline{210} \oplus \underline{126} \oplus \underline{\overline{126}} \oplus \underline{10}$:

$$\begin{aligned}(a,b,c,d,e) &= (-25, -57/4, 130, -355/4, -9/4), \\ (a',b') &= (26, -213/8), \end{aligned} \quad (20)$$

Case (II): $\underline{54} \oplus \underline{45} \oplus \underline{126} \oplus \underline{\overline{126}} \oplus \underline{10}$:

$$\begin{aligned}&(a,b,c,d,e) \\ &\quad = (-77/4, -45/4, 405/4, -135/2, -9/4), \\ &(a',b') = (81/4, -81/4), \end{aligned} \quad (21)$$

Case (III): $\underline{210} \oplus \underline{54} \oplus \underline{126} \oplus \underline{\overline{126}} \oplus \underline{10}$:

$$\begin{aligned}(a,b,c,d,e) &= (-109/4, -61/4, 565/4, -95, -9/4), \\ (a',b') &= (113/4, -57/2), \end{aligned} \quad (22)$$

Case (IV): $\underline{45} \oplus \underline{126} \oplus \underline{\overline{126}} \oplus \underline{10}$:

$$\begin{aligned}&(a,b,c,d,e) \\ &\quad = (-35/4, -31/6, 185/4, -385/12, -9/4), \\ &(a',b') = (37/4, -77/8). \end{aligned} \quad (23)$$

Although Cases (I) and (II) were derived in [20] some numerical and typographical errors have been corrected here, while Cases (III) and (IV) are new.

3.2 Gravitational corrections from $\dim = 5$ operators

In this subsection we derive gravitational corrections in Case (II) and Case (III) while such corrections in Case (I) were discussed in [20]. In addition to the renormalizable part of the Lagrangian of SUSY GUT, a $\dim = 5$ operator can be induced either in $\dim = 4$ gravity at the Planck scale ($M_C = M_{\rm Pl} = 2 \times 10^{18}$ GeV) or due to compactification of extra dimension(s) at scales $M_C = M_{\rm CS} \sim 10^{17}$ GeV [25]. We have

$$\mathcal{L}_{\rm gr} = -\frac{\eta}{2M_C} {\rm Tr}\left(F_{\mu\nu} \Sigma F^{\mu\nu}\right), \quad (24)$$

where, for example, $\Sigma \equiv \underline{210}, \underline{54} \subset SO(10)$ that contribute to the GUT symmetry breaking near the M_U and $M_C =$ compactification scale ($M_{\rm CS}$) of extra dimension(s), or the Planck scale ($M_{\rm Pl}$) in $\dim = 4$ gauge theory. When $\Sigma \equiv \underline{45} \subset SO(10)$ the contribution of the $\dim = 5$ operator in (24) identically vanishes. We will confine ourselves to the Cases (I)–(III) for gravitational corrections.

Although there are no exact theoretical constraint on η it could be positive or negative with plausible values up to $|\eta| \approx {\rm O}(10)$. Whereas $\underline{210}$ and $\underline{54}$ are present in Cases (I) and (II), respectively, both are present in Case (III). In [20] gravitational effects were derived only for Case (I) corresponding to $\Sigma \equiv \underline{210}$ with a normalization factor 1/8 instead of 1/2 as given in (24) [26]. In order to compare with gravitational corrections resulting from (24) with $\Sigma \equiv \underline{54}$ we evaluate them for Case (I) with the common normalization factor of 1/2. In a number of earlier investigations the effects of such operators on GUT predictions have been found to be quite significant [18,20, 25–27]. In the presence of $SO(10) \to G_{2213}$ such operators modify the GUT boundary condition on the coupling constants which has the general form at $\mu = M_U$ of

$$\begin{aligned}\alpha_{2L}(M_U)(1+\epsilon_{2L}) &= \alpha_{2R}(M_U)(1+\epsilon_{2R}) \\ &= \alpha_{BL}(M_U)(1+\epsilon_{BL}) = \alpha_{3C}(M_U)(1+\epsilon_{3C}) \\ &= \alpha_G(M_U). \end{aligned} \quad (25)$$

These boundary conditions lead to the corresponding gravitational corrections on the four gauge couplings,

$$\Delta_i^{\rm gr} = -\frac{\epsilon_i}{\alpha_G}, \; i = 2L, 2R, BL, 3C. \quad (26)$$

Then using the procedure of [20], analytic formulas for the gravitational corrections of the two mass scales are derived:

$$\begin{aligned}\left(\ln \frac{M_I}{M_Z}\right)_{\rm gr} &= \frac{2\pi(A'\epsilon' - A\epsilon'')}{\alpha_G(AB' - A'B)}, \\ \left(\ln \frac{M_U}{M_Z}\right)_{\rm gr} &= \frac{2\pi(B\epsilon'' - B'\epsilon')}{\alpha_G(AB' - A'B)}, \end{aligned} \quad (27)$$

where

$$\begin{aligned}B = B' &= \frac{5}{3} a_Y - \frac{2}{3} a'_{BL} - a'_{2R}, \\ A &= a'_{2R} + \frac{2}{3} a'_{BL} - \frac{5}{3} a'_{2L}, \\ A' &= a'_{2R} + \frac{2}{3} a'_{BL} + a'_{2L} - \frac{8}{3} a'_{3C}, \\ \epsilon'' &= \epsilon_{2L} + \epsilon_{2R} + \frac{2}{3}\epsilon_{BL} - \frac{8}{3}\epsilon_{3C}, \\ \epsilon' &= \epsilon_{2R} + \frac{2}{3}\epsilon_{BL} - \frac{5}{3}\epsilon_{2L}. \end{aligned} \quad (28)$$

We will need the numerical values of A, A', B, B' defined through (28) which are the same in all R-parity conserving cases with G_{2213}-intermediate gauge symmetry,

$$A = 40/3, \quad A' = 24,$$
$$B = B' = -4. \qquad (29)$$

With the generalized formulas given by (25)–(28) and the numerical values given in (29) we discuss specific gravitational corrections in three different cases as given below.

Case (I): $\underline{210} \oplus \underline{126} \oplus \underline{\overline{126}} \oplus \underline{10}$

In this case $\Sigma \equiv \underline{210}$, and we denote the unknown parameter in (24) as $\eta = \eta_1$. After taking into account a factor 4 in the normalization of the gauge kinetic term [26,27] and using an approximate relation between the GUT scale VEV ϕ_0 and the degenerate masses of superheavy gauge bosons, $M_U \approx (2/9)^{1/2} g_G \phi_0$, we have

$$\epsilon_{2R} = -\epsilon_{2L} = -\epsilon_{3C} = \frac{1}{2}\epsilon_{BL} = \epsilon_1,$$
$$\epsilon' = \epsilon_{2R} + \frac{2}{3}\epsilon_{BL} - \frac{5}{3}\epsilon_{2L} = 4\epsilon_1,$$
$$\epsilon'' = \epsilon_{2R} + \epsilon_{2R} + \frac{2}{3}\epsilon_{BL} - \frac{8}{3}\epsilon_{3C} = 4\epsilon_1, \qquad (30)$$

where

$$\epsilon_1 = \frac{3\eta_1}{4}\frac{M_U}{M_C}\frac{1}{\sqrt{4\pi\alpha_G}}. \qquad (31)$$

Using (29) and (30) in (27) gives

$$\left(\ln\frac{M_I}{M_Z}\right)_{\rm gr} = \frac{2\pi\epsilon_1}{\alpha_G},$$
$$\left(\ln\frac{M_U}{M_Z}\right)_{\rm gr} = 0, \qquad (32)$$

which were derived in [20] but with a different normalization factor for ϵ_1.

Case (II): $\underline{54} \oplus \underline{45} \oplus \underline{126} \oplus \underline{\overline{126}} \oplus \underline{10}$

In this case $\underline{45} \subset SO(10)$ does not contribute to the dim = 5 operator of (24). Using $\Sigma \equiv \underline{54}$ and denoting $\eta = \eta_2$ in (24), we derive

$$\epsilon_{3C} = \epsilon_{BL} = \epsilon_2,$$
$$\epsilon_{2L} = \epsilon_{2R} = -\frac{3}{2}\epsilon_2,$$
$$\epsilon' = \epsilon_{2R} + \frac{2}{3}\epsilon_{BL} - \frac{5}{3}\epsilon_{2L} = \frac{5}{3}\epsilon_2,$$
$$\epsilon'' = \epsilon_{2L} + \epsilon_{2R} + \frac{2}{3}\epsilon_{BL} - \frac{8}{3}\epsilon_{3C} = -5\epsilon_2, \qquad (33)$$

where

$$\epsilon_2 = \frac{3\eta_2}{4}\frac{M_U}{M_C}\frac{1}{\sqrt{15\pi\alpha_G}}. \qquad (34)$$

Using (29) and (33) and (34) in (27), we get

$$\left(\ln\frac{M_I}{M_Z}\right)_{\rm gr} = \frac{5\pi\epsilon_2}{\alpha_G},$$
$$\left(\ln\frac{M_U}{M_Z}\right)_{\rm gr} = \frac{5\pi\epsilon_2}{4\alpha_G}. \qquad (35)$$

Equation (35) has the implication that if we attempt to change the unification mass by 1 order purely by gravitational corrections, then the intermediate scale would change by approximately 4 orders.

Case (III): $\underline{210} \oplus \underline{54} \oplus \underline{126} \oplus \underline{\overline{126}} \oplus \underline{10}$

The importance of this case emphasizing the presence of $\underline{54}$ in addition to $\underline{210}$ for realistic SUSY $SO(10)$ breaking leading to Type II seesaw dominance for neutrino masses has been elucidated in [21] in the single step breaking case. In our case with G_{2213}-intermediate symmetry both $\underline{54}$ and $\underline{210}$ contribute separately to the dim = 5 operator with

$$\mathcal{L}_{\rm gr} = -\frac{\eta_1}{2M_C}{\rm Tr}(F_{\mu\nu}\phi_{210}F^{\mu\nu})$$
$$-\frac{\eta_2}{2M_C}{\rm Tr}(F_{\mu\nu}\phi_{54}F^{\mu\nu}). \qquad (36)$$

Then

$$\Delta_i^{\rm gr} = -(\epsilon_i^{54} + \epsilon_i^{210})/\alpha_G,$$
$$i = 2L, 2R, BL, 3C.$$

The relations (32) and (35) hold separately leading to

$$\left(\ln\frac{M_U}{M_Z}\right)_{\rm gr} = \frac{5\pi\epsilon_2}{4\alpha_G},$$
$$\left(\ln\frac{M_I}{M_Z}\right)_{\rm gr} = \frac{5\pi\epsilon_2}{\alpha_G} + \frac{2\pi\epsilon_1}{\alpha_G}.$$

Comparing (31) and (34) gives $\epsilon_1/\epsilon_2 = (15/4)^{1/2}\eta_1/\eta_2$. In the next section we use these results to study the effects of gravitational corrections on $SO(10)$ gauge coupling.

4 Perturbative $SO(10)$ gauge coupling at higher scales

In all the three cases the same lighter components contained in $\underline{126} \oplus \underline{\overline{126}} \oplus \underline{10}$ contribute to the one-loop and two-loop β-function coefficients below the GUT scale and none of the components in $\underline{210}, \underline{54}$, or $\underline{45}$ contribute to large runnings of the gauge couplings. Thus, ignoring threshold and gravitational corrections, the two-loop solution of RGEs is the same for all the four cases with

$$M_I^0 = 10^{15.2}\,{\rm GeV}, M_U^0 = 10^{16.11}\,{\rm GeV},$$
$$\alpha_G^0 = 0.043. \qquad (37)$$

Then adding threshold and gravitational corrections to the two-loop solutions the mass scales are expressed as

$$\ln\frac{M_U}{M_Z} = \ln\frac{M_U^0}{M_Z} + \Delta\ln\frac{M_U}{M_Z} + \left(\ln\frac{M_U}{M_Z}\right)_{\rm gr},$$
$$\ln\frac{M_I}{M_Z} = \ln\frac{M_I^0}{M_Z} + \Delta\ln\frac{M_I}{M_Z} + \left(\ln\frac{M_I}{M_Z}\right)_{\rm gr}. \quad (38)$$

When we include the corrections mentioned in Sect. 3 through (38), the resulting mass scales are modified in each case. The increased value of M_U then extends the range of perturbative$SO(10)$ gauge coupling up to the Planck scale. In what follows we discuss some examples of such solutions in each case.

The mass scales obtained including threshold corrections are denoted as $M_i^{(1)}$ and those obtained including both the threshold and gravitational corrections are denoted as $M_i^{(2)}$ $(i = I, U)$. We have

Case (I): As shown in Sect. 2 the lower bound on the unification mass in this case is 5.8×10^{17} GeV. Using threshold and gravitational corrections we examine how far this constraint can be satisfied. Using the effective mass parameters

$$M''_{2L} = M_U,\ M''_{3C} = 0.87M_U,$$
$$M''_{2R} = 1.5M_U, M''_{BL} = 1.8M_U,$$
$$M'_{1Y} = M_I,$$

we obtain, including only threshold effects,

$$M_U^{(1)} = 6.54 \times 10^{17}\,\text{GeV}, \quad M_I^{(1)} = 7 \times 10^{11}\,\text{GeV}.$$

Using this modified value, $M_U = M_U^{(1)} = 6.54 \times 10^{17}$ GeV, (6) gives the perturbative value of the GUT gauge coupling at $\Lambda = M_{\rm Pl}$ with $\alpha_G(M_{\rm Pl}) = 0.587$. The effects are more (less) prominent if the mass gap of the effective mass parameters are increased (decreased) for which the values of the corresponding gauge coupling will be smaller (larger). It is easily checked that the inequality (9) is satisfied. Since the gravitational corrections do not affect the GUT scale, but affect only the intermediate scale which is of the same order as the right-handed neutrino mass, in this case any desired value of the intermediate scale matching the scale of leptogenesis, or the Pecei–Quinn symmetry breaking scale, or even a value close to the minimal GUT scale can be obtained. Thus the model is potentially interesting from the point of view of neutrino physics, leptogenesis and strong CP-violation. Other examples of solutions for this case are shown in Table 2.

Case (II): As shown in Sect. 2 the value of $a_{\rm Higgs} = 91$ in this case gives the lower bound $M_U > 3 \times 10^{17}$ GeV. To examine how far threshold and gravitational corrections may allow for such high unification scales, at first we consider only threshold corrections. Using the effective mass parameters

$$M''_{2L} = M_U, M''_{2R} = 1.7M_U, M''_{BL} = 2M_U,$$
$$M''_{3C} = 0.87M_U, M'_{1Y} = M_I,$$

gives, including threshold corrections but ignoring gravitational corrections,

$$M_I^{(1)} = M_R = 2.91 \times 10^{11}\,\text{GeV},$$
$$M_U^{(1)} = 3.85 \times 10^{17}\,\text{GeV}.$$

Clearly the mass gaps near the GUT scale are reasonably small and are confined between $0.87M_U$ and $2M_U$. Now adding gravitational corrections with $\eta_2 = 3.0$ gives

$$M_I^{(2)} = 9.31 \times 10^{12}\,\text{GeV}, \quad M_U^{(2)} = 8.95 \times 10^{17}\,\text{GeV}.$$

Using the value of $M_U = M_U^{(2)} = 8.95 \times 10^{17}$ GeV we obtain from (6) the perturbative value of the gauge coupling , $\alpha_G(M_{\rm Pl}) = 0.084$. Evaluating the RHS of inequality (9) gives

$$a_{\rm Higgs} < 180.$$

Noting from Table 1 that for this case $a_{\rm Higgs} = 91$ it is clear that inequality (9) is satisfied ensuring perturbativity of $SO(10)$ gauge coupling up to the Planck scale. Another example of a solution including gravitational correction is given in Table 2.

Case (III): As shown in Sect. 2 for this case $a_{\rm Higgs} = 139$ and (9) gives the lower bound $M_U \geq 6.25 \times 10^{17}$ GeV to ensure perturbative gauge coupling up to the Planck scale. The necessity of both $\underline{210}$ and $\underline{54}$ for realistic SUSY $SO(10)$ breaking directly to MSSM has been emphasized in [21]. With G_{2213}-intermediate breaking this case appears to be interesting as it shows the possibility that dominant gravitational corrections with marginal or negligible threshold effects can elevate the GUT scale closer to the Planck scale [25]. Although, in principle, threshold effects are somewhat larger in this case compared to the Cases (I), (II) and (IV) because of the presence of an extended size of the Higgs representations, their actual values are controlled by the choice of the mass gap in the effective mass parameters. For example, using the effective mass parameters,

$$M''_{2L} = M_U,\ M''_{3C} = 0.87M_U,$$
$$M''_{2R} = 1.6M_U, M''_{BL} = 1.6M_U,$$
$$M'_{1Y} = M_I,$$

we obtain, including only threshold effects,

$$M_U^{(1)} = 7.57 \times 10^{17}\,\text{GeV}, \quad M_I^{(1)} = 3.92 \times 10^{11}\,\text{GeV}.$$

Then (6) gives the perturbative gauge coupling $\alpha_G(M_{\rm Pl}) \simeq 0.25$.

Further addition of gravitational corrections with $\eta_1 = -3.0$ and $\eta_2 = 5.0$ gives higher values of the unification scale closer to $M_{\rm Pl}$,

$$M_U^{(2)} = 2.95 \times 10^{18}\,\text{GeV}, \quad M_I^{(2)} = 6.96 \times 10^{12}\,\text{GeV}.$$

Using this high value of the unification scale $M_U = M_U^{(2)} = 2.95 \times 10^{18}$ GeV we obtain from (6) the perturbative value of the gauge coupling $\alpha_G(M_{\rm Pl}) \simeq 0.049$. We

Table 2. Perturbative$SO(10)$ gauge coupling at higher scales including threshold and gravitational corrections to two-loop solutions for which $M_I^0 = 10^{15.20}$ GeV and $M_U^0 = 10^{16.11}$ GeV. The mass scales $M_i^{(1)}$ $(i = I, U)$ have been obtained including threshold corrections and $M_i^{(2)}$ $(i = I, U)$ including both threshold and gravitational corrections

Higgs Rep.	Mass parameters	$M_I^{(1)}$(GeV)	$M_U^{(1)}$(GeV)	η_1	η_2	$M_I^{(2)}$(GeV)	$M_U^{(2)}$(GeV)	$\alpha_G(M_{\mathrm{Pl}})$
$\underline{210}\oplus$	$M'_{1Y} = M_I, M''_{2L} = M_U$			0.0	–	7×10^{11}	6.54×10^{17}	
$\underline{126} \oplus \underline{\overline{126}}$	$M'_{2R} = 1.5M_U, M''_{BL} = 1.8M_U$	7×10^{11}	6.54×10^{17}	0.5	–	1.08×10^{12}	6.54×10^{17}	0.587
$\oplus\underline{10}$	$M''_{3C} = 0.87M_U$			5.0	–	5.49×10^{13}	6.54×1017	
	$M'_{1Y} = 2M_I, M''_{2L} = 2.5M_U$			1.5	–	1.05×10^{13}	8.38×10^{17}	0.125
	$M'_{2R} = 4.5M_U, M''_{BL} = 3.76M_U$	2.84×10^{12}	8.38×10^{17}	4.9	–	2.05×10^{14}	8.38×10^{17}	
	$M''_{3C} = 2.1M_U$							
$\underline{54} \oplus \underline{45}\oplus$	$M'_{1Y} = M_I, M''_{2L} = M_U$			–	3.0	9.31×10^{12}	8.95×10^{17}	0.084
$\underline{126} \oplus \underline{\overline{126}}$	$M'_{2R} = 1.7M_U, M''_{BL} = 2M_U$	2.91×10^{11}	3.85×10^{17}					
$\oplus\underline{10}$	$M''_{3C} = M_U$			–	6.5	5.3×10^{14}	2.1×10^{18}	0.047
$\underline{210} \oplus \underline{54}\oplus$	$M'_{1Y} = M_I, M''_{2L} = M_U$			−1.5	2.5	1.65×10^{12}	1.5×10^{18}	0.09
$\underline{126} \oplus \underline{\overline{126}}$	$M'_{2R} = 1.6M_U, M''_{BL} = 1.6M_U$	3.92×10^{11}	7.57×10^{17}					
$\oplus\underline{10}$	$M''_{3C} = 0.87M_U$			−3.0	5.0	6.96×10^{12}	2.95×10^{18}	0.049
	$M'_{1Y} = M_I, M''_i = M_U$	3.3×10^{14}	1.43×10^{16}	−20.0	14.2	1.09×10^{14}	8.61×10^{17}	0.175
	i=2L, 2R, BL, 3C							
$\underline{45}\oplus$	$M'_{1Y} = M_I, M''_{2L} = 1.5M_U$							
$\underline{126} \oplus \underline{\overline{126}}$	$M'_{2R} = M''_{BL} = 3.5M_U$	1.2×10^{15}	4.73×10^{17}	–	–	–	–	0.10
$\oplus\underline{10}$	$M''_{3C} = M_U$							

also note that the perturbative inequality (9) is easily satisfied with $\Lambda = M_{\mathrm{Pl}}$. Another example of such solution for this case is shown in Table 2, where both threshold and gravitational corrections have been included.

Now we show that with negligible GUT threshold corrections, but with the inclusion of gravitational corrections alone in this case it is also possible to obtain high values of the unification scale and perturbative gauge coupling up to the Planck scale. For the sake of simplicity ignoring all high scale threshold corrections by choosing $M'_{1Y} = M_I$ and $M''_i = M_U$ $(i = 2L, 2R, BL, 3C)$ and using $\eta_1 = -20.0$ and $\eta_2 = 14.2$ leads to $\epsilon_1 = -0.128$ and $\epsilon_2 = 0.047$. Then (36) gives $\left(\ln \frac{M_U}{M_Z}\right)_{\mathrm{gr}} = 4.33$ and $\left(\ln \frac{M_I}{M_Z}\right)_{\mathrm{gr}} = -1.49$. When added to two-loop solutions including the weak scale SUSY threshold corrections, we obtain

$$M_U^{(2)} = 8.61 \times 10^{17}\,\mathrm{GeV}, \quad M_I^{(2)} = 1.09 \times 10^{14}\,\mathrm{GeV}.$$

Using $M_U = M_U^{(2)} = 8.61 \times 10^{17}$ GeV in (6) gives the perturbative value of the gauge coupling at $\Lambda = M_{\mathrm{Pl}}$ with $\alpha_G(M_{\mathrm{Pl}}) = 0.175$. We find that the RHS of (9) is $\simeq 190$ as compared to the value $a_{\mathrm{Higgs}} = 139$ for this case and the perturbative inequality is satisfied. Thus, including gravitational corrections alone the $SO(10)$ model with such a choice of Higgs representation guarantees perturbative SUSY $SO(10)$ gauge coupling up to the Planck scale.

Case (IV): As shown in Sect. 2, $a_{\mathrm{Higgs}} = 79$ through (9) gives the lower bound $M_U \geq 1.5 \times 10^{17}$ GeV in this case to ensure perturbative gauge coupling up to Planck scale. As there are no gravitational corrections due to the dim = 5 operator for this case we will consider only threshold corrections. Using

$$M''_{2L} = 1.5M_U, M''_{2R} = M''_{BL} = 3.5M_U,$$
$$M''_{3C} = M_U, M'_{1Y} = M_I,$$

we obtain

$$M_I^{(1)} = 1.2 \times 10^{15}\,\mathrm{GeV}, \qquad M_U^{(1)} = 4.7 \times 10^{17}\,\mathrm{GeV}.$$

Using $M_U = M_U^{(1)} = 4.7 \times 10^{17}$ GeV in (6) gives the perturbative gauge coupling at the Planck scale with $\alpha_G(M_{\mathrm{Pl}}) \simeq 0.10$. The RHS of (9) is found to be $\simeq 119$ and the inequality is satisfied.

5 Proton lifetime predictions

As pointed out in Sect. 1, the experimental lower limit on the proton lifetime for the decay mode $p \to e^+\pi^0$ mediated by superheavy gauge bosons or equivalently through the effective dim = 6 operator sets a lower limit on the GUT scale, $M_U \geq 5.6 \times 10^{15}$ GeV which is easily satisfied in the supergrand-desert scenario for which, excluding threshold or gravitational corrections, $M_U = 2 \times 10^{16}$ GeV. The lower bounds on M_U obtained in Sect. 2 for Cases (I)–(IV), purely from the requirement of perturbativity of the $SO(10)$ gauge coupling up to the Planck scale, are found to be satified by the RG solutions for the mass scales when threshold corrections, or gravitational corrections, or both are included in the intermediate scale models. In Case (IV) for which the Higgs representations $\underline{45} \oplus \underline{126} \oplus \overline{\underline{126}} \oplus \underline{10}$ have the smallest size among all the four cases, the solutions of the RGEs for the mass scales are consistent with the lower bound $M_U \geq 1.5 \times 10^{17}$ GeV when threshold effects are included. In each of the four cases the corresponding lower bound on the unification scale translates into a lower bound on the proton lifetime. The shortest of these lower bounds on the proton lifetime occurs in the Case (IV),

$$\tau \left(p \to e^+\pi^0\right) \geq 2.1 \times 10^{39}\ \text{years}. \tag{39}$$

In Cases (I)–(III) the lifetimes are longer than this value as can be approximately estimated using Table 2. These analyses suggest that the decay mode $p \to e^+\pi^0$ which has a lifetime at least 6 orders longer than the current limit is inaccessible to experimental observation.

Supersymmetric decay modes of the proton such as $p \to \mathrm{K}^+\bar{\nu_\mu}$, $p \to \mathrm{K}^+\bar{\nu_\tau}$ and other ones are characteristic predictions in SUSY GUTs [30]. These decays are mediated by higgsinos ($T_{\tilde{C}}$) which are superpartners of color triplet Higgs scalars (T_C) having superheavy masses near the GUT scale. As pointed out the experimental lower limit on the proton lifetime given in (2) sets the lower bound on the superheavy color triplet higgsino mass, $M_{T_{\tilde{C}}} \geq 10^{17}$ GeV.

In SUSY $SU(5)$ there is one such pair of higgsinos which are superpartners of Higgs color triplets contained in $\underline{5} \oplus \overline{\underline{5}} \subset$ SU(5); in SUSY $SO(10)$ models the color triplet Higgs may be treated as a linear combination of the triplets contained in $\underline{10}$, $\underline{126} \oplus \overline{\underline{126}}$ and $\underline{45}$, or $\underline{54}$, or $\underline{210}$ depending upon the choice of specific Higgs representations used to break the GUT symmetry to G_{2213} [19]. For the sake of simplicity we ignore finer details of the calculations and give plausibility arguments to show that for these decays governed by the effective dim = 5 operators, proton lifetimes ranging from the present experimental limit to several orders longer can be a natural prediction of the intermediate breaking scenario.

In a supergrand-desert model like SUSY $SU(5)$, the constraint on the color triplet higgsino mass is obtained using the unification condition including threshold corrections: $g_G(\Lambda_U) = g_{1Y}(\Lambda_U) + \Delta_{1Y}(\Lambda_U) = g_{2L}(\Lambda_U) + \Delta_{2L}(\Lambda_U)$, where g_G = GUT gauge coupling and Λ_U is the GUT scale. This leads to the constraint $g_G^{-2}(\Lambda_U) - g_{3C}^{-2}(\Lambda_U) = (3/20\pi^2)\ln(M_{T_{\tilde{C}}}/\Lambda_U)$ and $M_{T_{\tilde{C}}} \simeq$ few $\times\ 10^{15}$ GeV [33]. However, including gravitational corrections a large increase of the higgsino mass even up to 4 orders of magnitude has been suggested in SUSY $SU(5)$ [18].

But in the presence of G_{2213}-intermediate symmetry in the mass range $\mu = M_I - M_U$, the GUT scale constraint equating g_{1Y} and g_{2L} is absent since the $U(1)_Y$ gauge coupling splits above the scale M_I into two separate unconstrained gauge couplings,

$$\frac{1}{g_{1Y}^2(\mu)} = \frac{2}{5}\frac{1}{g_{BL}^2(\mu)} + \frac{3}{5}\frac{1}{g_{2R}^2(\mu)},\ \mu = M_I - M_U.$$

As the gauge symmetry near Λ_U is no longer the SM, but G_{2213}, the simple SU(5) relation among g_G, g_{3C} and $M_{T_{\tilde{C}}}$ is no longer valid. Further, unlike $SU(5)$ where the Higgs color triplet and anti-triplet are confined to its Higgs representations, $\underline{5} \oplus \overline{\underline{5}}$, in $SO(10)$ their number is much larger, as they can originate from Higgs representations like $\underline{10}$, $\underline{126} \oplus \overline{\underline{126}}$, $\underline{45}$, $\underline{54}$, and $\underline{210}$. In view of these there is no similar precision constraint on $M_{T_{\tilde{C}}}$ as in SUSY $SU(5)$ originating from gauge coupling unification. In the presence of such a two-step breaking through G_{2213}-intermediate gauge symmetry the value of $M_{T_{\tilde{C}}}$ can easily exceed 10^{17} GeV.

Since our lower bounds needed for perturbative gauge coupling up to the Planck scale as shown in Sect. 4 are in the range

$$M_U \geq (1.5\text{–}6.2) \times 10^{17}\ \text{GeV},$$

and the lifetimes for the supersymmetric decay modes are proportional to $M_{T_{\tilde{C}}}^2$, the lower bound on lifetimes are expected to be longer by factors ranging between 2.2 and 38 compared to the single step breaking scenario. This is due to the natural expectation that without additional fine tuning all superheavy components including the color triplets would have masses close to M_U. Thus, the criteria of perturbative gauge coupling up to the Planck scale, which are easily met by threshold or gravitational corrections in the four cases of R-parity conserving SUSY $SO(10)$, constrain the unification scales with $M_U \geq$ (1.5–6.2) $\times 10^{17}$ GeV which in turn predict for the supersymmetric decay modes of the proton,

$$\tau(p \to K^+\overline{\nu}_\tau) \geq (2\text{–}9) \times 10^{34}\ \text{years}. \tag{40}$$

But it is well known that even without additional fine tuning the superheavy components could easily be a few times lighter or heavier than M_U. Stretching this factor to the value of $\simeq 1/6$ or 6 the lower limit on the proton lifetime has a wider range starting from the current experimental limit up to a value which is 2–3 orders longer.

It is interesting to note that the high scale perturbative renormalization group relations (6) or (9) and the R-parity conservation in SUSY $SO(10)$ predict these lower bounds on the unification scales, the smallest one being $M_U \simeq 1.5 \times 10^{17}$ GeV. The resulting longer values of the proton lifetime predictions are consequences of generalized perturbative criteria in R-parity conserving SUSY $SO(10)$ which are also solutions to perturbative renormalization group equations including threshold or gravitational corrections.

6 Summary and conclusion

SUSY $SO(10)$ with $\underline{126} \oplus \overline{\underline{126}}$ and other Higgs representations in the case of single step breaking to MSSM has many attractive features for all fermion masses and mixings while ensuring R-parity conservation. But the popular argument raised against the model is that it violates perturbative gauge theory as the GUT coupling blows off even at mass scales a few times larger than the conventional GUT scale. In this paper we have shown that the requirement that the GUT gauge coupling remains perturbative up to the Planck scale imposes lower bounds on the unification scale which are at least 1 order larger than the conventional GUT scale. We have shown that the solutions to RGEs respecting these lower bounds are in fact possible if the threshold and/or gravitational corrections are included. The four different models discussed here ensure perturbative gauge coupling at least up to the Planck scale. The proton lifetime for $p \to e^+\pi^0$ becomes longer at least by nearly 6 orders of magnitude compared to the current experimental limit. For the supersymmetric decay modes a wide range of lifetimes is possible extending from the current experimental limit up to values 2–3 orders longer. These consequences follow without any additional fine tuning and by adopting the plausible criteria, that is: in the presence of the intermediate gauge symmetry, all superheavy masses including the color triplet.

Higgsinos have masses similar to the new high values of the unification scales. Although we have used the value of reduced Planck scale for this analysis, we have checked that our method also works with $M_{\rm Pl} = 1.2 \times 10^{19}$ GeV as defined by the Particle Data Group.

Due to high values of the intermediate scale, the success of the explanation of fermion masses and mixings is expected to be similar to the single step breaking case, but the additional advantages of a high unification scale is that it ensures perturbative SUSY $SO(10)$ with R-parity conservation at least up to the Planck scale and increases the stability of the proton. A different scenario for the increase of the proton stability in single step breaking of SUSY $SO(10)$ with R-parity conservation has been suggested recently by introducing specific textures [19] where the perturbative condition on SUSY $SO(10)$ gauge coupling holds up to $\mu = \text{few} \times 2 \times 10^{16}$ GeV.

Acknowledgements. The work of M.K.P. is supported by the project No. SP/S2/K-30/98 of Department of Science and Technology, Govt. of India.

References

1. E. Witten, Phys. Lett. B **105**, 267 (1981); R.K. Kaul, Phys. Lett. B **109**, 19 (1982); S. Dimopoulos, H. Georgi, Nucl. Phys. B **193**, 150 (1981); Phys. Lett. B **117**, 287 (1982); S. Dimopoulos, S. Raby, F. Wilczek, Phys. Rev. D **24**, 1681 (1981)
2. P. Minkowski, Phys. Lett. B **67**, 421 (1977); M. Gell-Mann, P. Ramond, R. Slansky, in Supergravity, Proceedings of the Workshop, Stony Brook, New York, 1997, edited by P. van Niewenhuizen, D.Z. Freedman (North-Holland, Amsterdam 1979); T. Yanagida, in Proceedings of the Workshop on Unified Theory and Baryon Number in the Universe, Tsukuba, Japan, 1997, edited by O. Sawada, A. Sugimoto (KEK, Japan 1979) p. 95; S.L. Glashow, The future of elementary particle physics, in Proceedings of the 1979 Cargese Summer Institute on Quarks and Leptons, edited by M. Levy et al. (Plenum Press, New York 1980), p. 687; R.N. Mohapatra, G. Senjanovic, Phys. Rev. Lett. **44**, 912 (1980)
3. R.N. Mohapatra, G. Senjanovic, Phys. Rev. D **23**, 165 (1981)
4. S.M. Barr, Phys. Rev. Lett. **92**, 101601 (2004); Phys. Rev. D **70**, 075005 (2004)
5. P. Langacker, M. Luo, Phys. Rev. D **44**, 817 (1991); J. Ellis, S. Kelley, D.V. Nanopoulos, Phys. Lett. B **249**, 441 (1990); U. Amaldi, W. de Boer, H. Furstanau, Phys. Lett. B **260**, 447 (1991); F. Anselmo, L. Cifarelli, A. Peterman, A. Zichichi, Nuovo Cimento A **104**, 1827 (1991); I. Antoniadis, J. Ellis, S. Kelley, D.V. Nanopoulos, Phys. Lett. B **272**, 31 (1991); U. Amaldi, W. de Boer, P.H. Frampton, H. Furstanau, J.T. Liu, Phys. Lett. B **281**, 374 (1992)
6. J.C. Pati, A. Salam, Phys. Rev. Lett. **31**, 661 (1973); Phys. Rev. D **10**, 275 (1974)
7. H. Georgi, S.L. Glashow, Phys. Rev. Lett. **32**, 438 (1974)
8. H. Georgi, Particles and Fields, edited by C.E. Carlson (AIP, New York 1975); H. Frizsch, P. Minkowski, Ann. Phys. **93**, 193 (1975)
9. F. Gursey, P. Ramond, P. Sikivie, Phys. Lett. B **60**, 177 (1976); Y. Achiman, B. Stech, Phys. Lett. B **77**, 398 (1978)
10. R.N. Mohapatra, J.C. Pati, Phys. Rev. D **11**, 566 (1975); R.N. Mohapatra, G. Senjanovic Phys. Rev. D **12**, 1502 (1975)
11. M.K. Parida, P.K. Patra, Phys. Rev. Lett. **66**, 858 (1991); **68**, 768 (1992); M.K. Parida, Phys. Rev. D **57**, 2736 (1998); C.S. Aulakh, A. Melfo, A. Rasin, G. Senjanovic, Phys. Rev. D **58**, 115007 (1998); C.S. Aulakh, A. Melfo, G. Senjanovic, Phys. Rev. D **57**, 4174 (1998); C.S. Aulakh, B. Bajc, A. Melfo, A. Rasin, G. Senjanovic, Phys. Lett. B **460**, 325 (1999)
12. K.S. Babu, R.N. Mohapatra, Phys. Rev. Lett. **70**, 2845 (1993)
13. B. Bajc, G. Senjanovic, F. Vissani, Phys. Rev. Lett. **90**, 051802 (2003) [hep-ph/0210207]; Phys. Rev. D **70**, 093002 (2004) [hep-ph/0402140]; B. Bajc, AS. Melfo, G. Senjanovic, F. Vissani, Phys. Rev. D **70**, 035007 (2004)
14. H.S. Goh, R.N. Mohapatra, S.P. Ng, Phys. Lett. B **570**, 215 (2003); Phys. Rev. D **68**, 115008 (2003); B. Dutta, Y. Mimura, R.N. Mohapatra, Phys. Rev. D **69**, 115014 (20004); B. Dutta, Y. Mimura, R.N. Mohapatra, hep-ph/0406262; S. Bertolini, M. Frigerio, M. Malinsky, Phys. Rev. D **70**, 095002 (20004)
15. K.S. Babu, C. Macesanu, hep-ph/0505200
16. H. Murayama, A. Pierce, Phys. Rev. D **65**, 055009 (2002)
17. B. Bajc, P.F. Perez, G. Senjanovic, Phys. Rev. D **66**, 075005 (20002); B. Bajc, A. Melfo, G. Senjanovic, hep-ph/0304051

18. K. Tobe, J.D. Wells, hep-ph/0312159
19. B. Dutta, Y. Miura, R.N. Mohapatra, Phys. Rev. Lett. **94**, 091804 (2005)
20. M.K. Parida, B. Purkayastha, C.R. Das, B.D. Cajee, Eur. Phys. J. C **28**, 353 (2003)
21. H.S. Goh, R.N. Mohapatra, S. Nasri, Phys. Rev. D **70**, 075022 (2004) [hep-ph/0408139]
22. D. Chang, T. Fukuyama, Y. Keun, T. Kikuchi, N. Okada, hep-ph/0412011
23. D. Chang, R.N. Mohapatra, M.K. Parida, Phys. Rev. Lett. **52**, 1052 (1984); Phys. Rev. D **30**, 1052 (1984); D. Chang, R.N. Mohapatra, J. Gipson, R.E. Marshak, M.K. Parida, Phys. Rev. D **31**, 1718 (1985)
24. S. Weinberg, Phys. Lett. B **91**, 51 (1980); L. Hall, Nucl. Phys. B **179**, 75 (1981); M.K. Parida, Phys. Lett. B **196**, 163 (1987); L. Lavoura, L. Wolfenstein, Phys. Rev. D **48**, 264 (1993); R.N. Mohapatra, M.K. Parida, Phys. Rev. D **47**, 264 (1993); Dae-Gyu Lee, R.N. Mohapatra, M.K. Parida, M. Rani, Phys. Rev. D **51**, 229 (1995)
25. Q. Shafi, C. Wetterich, Phys. Rev. Lett. **52**, 875 (1984); C.T. Hill, Phys. Lett B **135**, 47 (1984); M. Drees, Phys. Lett. B **158**, 409 (1985); Phys. Rev. D **33**, 1468 (1986); J.R. Ellis, K. Enqvist, D.V. Nanopoulos, K. Tamvakis, Phys. Lett. B **155**, 381 (1985); M.K. Parida, P.K. Patra, A.K. Mohanty, Phys. Rev. D **39**, 316 (1989); L.J. Hall, U. Sarid, Phys. Rev. Lett. **70**, 2673 (1993); K. Huitu, Y. Kamamura, T. Kobayashi, K. Puolamaki, Phys. Lett. B **468**, 111 (1999); P.G.O. Freund, Nucl. Phys. B **209**, 146 (1982)
26. M.K. Parida, P.K. Patra, Phys. Lett. B **234**, 45 (1990)
27. M.K. Parida, P.K. Patra, Phys. Rev. D **39**, 2000 (1989); D **44**, 2179 (1991)
28. M. Carena, S. Pokorski, C.E.M. Wagner, Nucl. Phys. B **406**, 59 (1993)
29. P. Langacker, N. Polonsky, Phys. Rev. D **47**, 4028 (1993)
30. K.S. Babu, J.C. Pati, F. Wilczek, Nucl. Phys. B **566**, 33 (2000) [hep-ph/9812538]
31. C.S. Aulakh, B. Bajc, G. Melfo, G. Senjanovic, F. Vissani, Phys. Lett. B **588**, 196 (2004) [hep-ph/0306242]
32. I. Antoniadis, C. Kounnas, K. Tamvakis, Phys. Lett. B **119**, 377 (1982)
33. J. Hisano, H. Murayama, T. Yanagida, Nucl. Phys. B **402**, 46 (1993); G. Dvali, S. Pokorski, Phys. Lett. B **379**, 126 (1996)